SPATIAL FREE CONVECTION OF NONLINEARLY VISCOUS FLUIDS AROUND AXISYMMETRIC BODIES

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The influence of the transverse curvature of axisymmetric bodies and the concentration factor on the friction and heat and mass exchange under the conditions of free spatial convection of nonlinearly viscous fluids is determined from the solution of nonlinear boundary value problems.

Free convection in rheologically complex media has attracted considerable attention in recent years [1, 2]. It has been shown in a number of experimental researches [3, 4] that many flowing media exhibit nonlinearly viscous properties for the range of shear velocities realizable under free convection conditions, and can be described by a "power-law" rheological equation of state. Plane problems of the free convection of a non-Newtonian fluid have been examined in [1–4], while the Stewart paper on the spatial problem should be noted [5]. However, spatial problems of the free convection of non-Newtonian fluids around slender bodies of revolution (the boundary layer thickness is commensurate with the radius of the body of revolution) have not generally been considered up to now. Such problems are often encountered in applications, especially in probe measurement techniques and the remote control of technological processes. Analogous problems of dynamics and heat exchange have been considered in a number of papers, of which [6, 7] can be noted, for forced convection conditions.

The dimensionless equations of a three-dimensional stationary boundary layer for the problem of thermal free convection of a nonlinearly viscous fluid around a vertical slender body of revolution, neglecting the temperature dependence of the physical properties (except the density in the "lift" term in the equation of motion) are

$$\mathrm{Pr}^{-\frac{2(n+1)}{3n+1}}\left(ru\frac{\partial u}{\partial x}+rv\frac{\partial u}{\partial y}\right)=\left[r\left|\frac{\partial u}{\partial y}\right|^{n-1}\frac{\partial u}{\partial y}\right]+r\Theta\cos\alpha_1, \tag{1}$$

$$\frac{\partial}{\partial x}(ru)+\frac{\partial}{\partial y}(rv)=0, \tag{2}$$

$$ru\frac{\partial\Theta}{\partial x}+rv\frac{\partial\Theta}{\partial y}=\frac{\partial}{\partial y}\left(r\frac{\partial\Theta}{\partial y}\right) \tag{3}$$

with the boundary conditions

$$u=v=0,\ \Theta=1 \quad \text{at}\ y=0,$$
$$u\to 0,\ \Theta\to 0 \quad \text{at}\ y\to\infty, \tag{4}$$

where

$$x=\frac{x'}{L},\quad y=\frac{y'}{L}\,\mathrm{Gr}^{\frac{1}{2(n+1)}}\mathrm{Pr}^{\frac{n}{3n+1}}, \tag{5}$$

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$$u = u' \Pr^{\frac{n+1}{3n+1}} [Lg\beta(T_0 - T_\infty)]^{-\frac{1}{2}},\ \Theta = \frac{T - T_\infty}{T_0 - T_\infty},$$
$$v = v' \Pr^{\frac{2n+1}{3n+1}} \mathrm{Gr}^{\frac{1}{2(n+1)}} [Lg\beta(T_0 - T_\infty)]^{-\frac{1}{2}}. \qquad (5)$$

The generalized Pr number is quite large, as a rule, for trickling highly viscous fluids. Hence, the thermal boundary layer thickness is much less than the dynamic boundary layer thickness. Consequently, the contribution of the inertial terms within the thermal boundary layer is negligible [1, 5]. Moreover, for slender bodies of revolution $\alpha_1 \approx 0$, hence, $\alpha_1 \approx 1$ can be assumed in the equations of motion (1). Then

$$\frac{\partial}{\partial y}\left[r\left|\frac{\partial u}{\partial y}\right|^{n-1}\frac{\partial u}{\partial y}\right] + r\Theta = 0, \qquad (6)$$

$$\frac{\partial}{\partial x}(ru) + \frac{\partial}{\partial y}(rv) = 0, \qquad (7)$$

$$ru\frac{\partial\Theta}{\partial x} + rv\frac{\partial\Theta}{\partial y} = \frac{\partial}{\partial y}\left(r\frac{\partial\Theta}{\partial y}\right). \qquad (8)$$

The boundary conditions (4) are hence modified to the form [2, 5]

$$u = v = 0,\ \Theta = 1 \quad \text{at} \quad y = 0;$$
$$\frac{\partial u}{\partial y} \to 0,\ \Theta \to 0 \quad \text{as} \quad y \to \infty. \qquad (9)$$

After having introduced the new dependent and independent variables and parameters

$$\xi = \int_0^x U^\alpha(x)\, r_0^\beta(x)\, dx, \qquad (10)$$

$$\eta = a\xi^{-\gamma} U(x) \int_0^y r\, dy, \qquad (11)$$

$$\Psi = b\xi^\gamma f(\eta), \qquad (12)$$

$$A = \frac{2\xi^\gamma}{aU(x)\, r_0^2(x)}, \qquad (13)$$

where Ψ is determined from the continuity equation $ru = \partial\Psi/\partial y$, $rv = -\partial\Psi/\partial x$, the system (6)-(9) is converted into the two equations

$$nf'''|f''|^{n-1}(1 + A\eta)^{\frac{n+1}{2}} + \frac{n+1}{2} f''|f''|^{n-1} \times A(1 + A\eta)^{\frac{n-1}{2}} + \Theta\xi^{\gamma(n+1)} r_0^{-(n+1)} a^{-(2n+1)} b^{-n} U^{-(2n+1)} = 0, \qquad (14)$$

$$\Theta''(1 + A\eta) + \Theta' A + \frac{b}{a}\Theta' f U^{\alpha-1} \xi^{2\gamma-1} r_0^{\beta-2} = 0 \qquad (15)$$

with the boundary conditions

$$f' = f = 0,\ \Theta = 1 \quad \text{at} \quad \eta = 0,$$
$$f'' \to 0,\ \Theta \to 0 \quad \text{at} \quad \eta \to \infty. \qquad (16)$$

The following functional relations

$$\xi^\gamma \sim U r_0^2,\quad \xi^{\gamma(n+1)} \sim U^{2n+1} r_0^{n+1},\quad \xi^{1-2\gamma} \sim r_0^{\beta-2} U^{\alpha-1}. \qquad (17)$$

result from the requirement of self-similarity of the problem. We obtain

$$\xi \sim U^{\frac{3n+1}{(n+1)\gamma}},\quad r_0 \sim U^{\frac{n}{n+1}}. \qquad (18)$$

from the first two relationships of the system (17). Substituting (18) into the last expresssion of the system (17) yields

$$U^{n\beta\gamma - 3n - 1 + 3n\gamma + \gamma + \alpha\gamma n + \alpha\gamma} \sim \text{const}, \qquad (19)$$

from which there must follow

$$n\beta\gamma - 3n - 1 + 3n\gamma + \gamma + \alpha\gamma n + \alpha\gamma = 0.$$

The relationship (19) is the single connection between the unknowns α, β, γ which can be determined from the system (17).

Going over to the physical variables, by using (10) and (17), yields the second equation

$$4n\gamma + 5\alpha n\gamma - 3\alpha n\gamma^2 - 3n - 1 + n\beta\gamma + 2\alpha\gamma + \gamma - \alpha\beta n\gamma^2 - \gamma^2\alpha^2 n - \alpha^2\gamma^2 - \alpha\gamma^2 = 0. \quad (20)$$

Moreover, the functional relationship

$$r_0 \sim x^{\frac{n\gamma}{3n+1-\gamma\beta n-\gamma\alpha n-\alpha\gamma}} \quad (21)$$

is obtained.

The system of two nonlinear algebraic equations (19), (20) in the three unknowns α, β, γ admits of a unique solution for α:

$$\alpha = -1 \quad (22)$$

and a set of solutions for β and γ. However, there must exist a connection between β and γ

$$\frac{1}{\gamma(\beta+2)} = \frac{n}{3n+1}. \quad (23)$$

Equations (21)-(23) turn out to be sufficient for the determination of the self-similarity conditions

$$U = x^{\frac{n+1}{3n+1}}, \quad r_0 = cx^{\frac{n}{3n+1}}. \quad (24)$$

In order to eliminate the constants from the system (14)-(15), it is necessary to demand

$$a = c^{\beta\gamma-1}\gamma^{\gamma}, \quad b = c^{1-\beta\gamma}\gamma^{-\gamma}. \quad (25)$$

It should be noted that according to (25), the parameters a and b take on distinct values depending on the selection of specific values of β and γ satisfying (23).

Therefore, the problem reduces to a system of nonlinear ordinary differential equations

$$nf'''|f''|^{n-1}(1+A\eta)^{\frac{n+1}{2}} + \frac{n+1}{2}f''|f''|^{n-1} \times A(1+A\eta)^{\frac{n-1}{2}} + \Theta = 0, \quad (26)$$

$$\Theta''(1+A\eta) + A\Theta' + f\Theta' = 0 \quad (27)$$

with the boundary conditions (16), wherein besides the rheological parameter n, an additional parameter A is contained which reflects the influence of the transverse curvature. The local heat exchange and friction coefficients are determined from the formulas

$$\mathrm{Nu} = -\Theta'(0)\,\mathrm{Gr}^{\frac{1}{2(n+1)}}\,\mathrm{Pr}^{\frac{n}{3n+1}}\,x^{\frac{2n+1}{3n+1}},$$

$$c_f = 2[f''(0)]^n\,\mathrm{Gr}^{-\frac{1}{2(n+1)}}\,\mathrm{Pr}^{\frac{n+2}{3n+1}}\,x^{-\frac{n}{3n+1}}. \quad (28)$$

The average coefficients equal, respectively,

$$\overline{\mathrm{Nu}} = -\frac{3n+1}{5n+2}\Theta'(0)\,\mathrm{Gr}^{\frac{1}{2(n+1)}}\,\mathrm{Pr}^{\frac{n}{3n+1}},$$

$$\overline{c_f} = \frac{2(3n+1)}{3n+2}[f''(0)]^n\,\mathrm{Gr}^{-\frac{1}{2(n+1)}}\,\mathrm{Pr}^{\frac{n+2}{3n+1}}. \quad (29)$$

The solution of this problem has been obtained on the Minsk-22 electronic computer by a modified Newton method. Some results of the computation are presented in Fig. 1a. The transverse curvature parameter affects $-\Theta'(0)$ and $f''(0)$ quite strongly, and therefore, also affects the heat exchange and friction coefficients of slender bodies of revolution. The weak dependence of $-\Theta'(0)$ on the non-Newtonian behavior parameter should be noted. Moreover, if magnification of the non-Newtonian properties results in a diminution in the quantity $-\Theta'(0)$ for small values of the curvature parameter, then the opposite tendency (Fig. 1a) is observed for large values of A. A diminution in the parameter n results in a sharper dependence of $f''(0)$ on the values of A. The quantity $f''(0)$ itself depends essentially on the index n of non-Newtonian behavior, and the more strongly, the greater the parameter A.

Thus, in those cases when the boundary layer thickness is commensurate with the radius of the body of revolution, the influence of the transverse curvature on the heat exchange and flow processes is quite significant.

Free convection can be caused not only by a thermodynamic factor (the gradient of the temperature field), but also by a chemical factor (the gradient of the concentration field). Hence, a simultaneous analysis of the influence of these factors on the free convection process is of interest. The equations of a two-component spatial boundary layer for the free convection problem of a non-Newtonian fluid around a slender body of revolution in the approximation used in [8] and neglecting thermal diffusion and diffusion heat conduction differ from the system (1)-(3) by the presence of the term $r\Theta_2 \cos\alpha$ due to the concentration lift, in the right side of (1) and by an additional equation for the diffusion of one of the components

$$ru\frac{\partial\Theta_2}{\partial x} + rv\frac{\partial\Theta_2}{\partial y} = \frac{\mathrm{Pr}_1}{\mathrm{Pr}_2}\frac{\partial}{\partial y}\left(r\frac{\partial\Theta_2}{\partial y}\right).$$

The boundary conditions are

$$u = v = 0,\ \Theta_1 = \Theta_2 = 1 \quad \text{at} \quad y = 0;$$
$$u \to 0,\ \Theta_1 \to 0,\ \Theta_2 \to 0 \quad \text{at} \quad y \to \infty.$$

When the boundary layer thickness is much less than the radius of the body of revolution ($\delta / r_0 \ll 1$), the system (1)-(3) is modified in such a way that the quantity r enters only in the continuity equation (2), but not in (3) and (4). The case of a slender body of revolution is considered henceforth. The case $\delta / r_0 \ll 1$ is considered at the end of the paper.

Performing manipulations analogous to those elucidated above yields the system of equations

$$nf'''|f''|^{n-1}(1+A\eta)^{\frac{n+1}{2}} + \frac{n+1}{2}f''|f''|^{n-1}A(1+A\eta)^{\frac{n-1}{2}} + \Theta_1 + K\Theta_2 = 0, \tag{30}$$

$$\Theta_1''(1+A\eta) + A\Theta_1' + \frac{f|f'|}{f'}\Theta_1' = 0, \tag{31}$$

$$\Theta_2''(1+A\eta) + A\Theta_2' + \frac{\mathrm{Pr}_2}{\mathrm{Pr}_1}\frac{f|f'|}{f'}\Theta_2' = 0 \tag{32}$$

with the boundary conditions

$$\begin{gathered} f' = f = 0,\ \Theta_1 = \Theta_2 = 1 \quad \text{for} \quad \eta = 0, \\ \Theta_1 \to 0,\ \Theta_2 \to 0,\ f'' \to 0 \quad \text{for} \quad \eta \to \infty, \end{gathered} \tag{33}$$

where

$$K = \mathrm{sign}(C_0 - C_\infty)\left[\mathrm{sign}(C_0 - C_\infty)\frac{\mathrm{Gr}_2}{\mathrm{Gr}_1}\right]^{\frac{1}{2-n}}. \tag{34}$$

It is hence necessary to replace x by $|x|$ in the variables (10)-(12) because the motion can be directed upward ($x > 0$, $f' > 0$, $u > 0$) or downward ($x < 0$, $f' < 0$, $u < 0$) depending on Pr_1, Pr_2, and K. The heat exchange and friction coefficients are determined by means of (28) and (29) and the local mass-exchange coefficients by

$$\mathrm{Nu}_2 = -\Theta_2'(0)\ \mathrm{Gr}_1^{\frac{1}{2(n+1)}}\mathrm{Pr}_1^{\frac{n}{3n+1}}x^{\frac{2n+1}{3n+1}}. \tag{35}$$

The following picture of the motion results from physical considerations. If $\mathrm{Pr}_1 \neq \mathrm{Pr}_2$, then if the signs of the temperature and concentration lifts agree the medium moves to one side in the whole flow domain (the sign of $f'(\eta)$ and therefore of u agrees with the sign of the lift). When the signs of the lift are distinct, namely, if one lift dominates in the whole domain, then just flow retardation holds; as the second lift grows the flow at the wall is slowed down still more, until there is a zone of oppositely directed flow near the surface which gradually extends into the whole flow domain as this moving force grows further.

If $Pr_1 = Pr_2$, no oppositely directed flows originate since the thermal and concentration boundary layer thicknesses are equal. When $K > -1$, the medium moves upward, while oppositely directed flows hold for $K < -1$ and, finally, the flow is at rest for $K = -1$.

The system (30)-(33) was solved numerically on a Minsk-22 electronic computer for $Pr_1 = Pr_2$. Some results of the computations are present in Fig. 1b and c. The heat and mass exchange, as well as the friction, depend substantially on the ratio Gr_2/Gr_1 and the curvature parameter. As the curvature parameter increases, the quantity $-\Theta'_{1,2}(0)$ grows. For diverse values of the ratio Gr_2/Gr_1 the curves of the dependences are almost equidistant (Fig. 1b). This equal distance is retained within each family constructed according to the non-Newtonian behavior parameter. In contrast to $-\Theta'_{1,2}(0)$, the quantity $f''(0)$ depends quite strongly on the flow index n (Fig. 1c).

When the boundary layer thickness is much less than the radius of the body of revolution ($\delta/r_0 \ll 1$), the spatial axisymmetric problem is reduced to the system (30)-(32) (the A = 0 case) by introduction of the variables

$$\eta = y\left[\frac{3n+1}{2n+1}\left(\frac{1}{r_0^n \cos\alpha_1}\right)^{\frac{3n+1}{n(2n+1)}}\int_0^X (\cos\alpha_1 r_0^n)^{\frac{1}{2n+1}}\,dX\right]^{-\frac{n}{3n+1}}, \tag{36}$$

$$u = (\cos\alpha_1)^{\frac{1}{n}}\left[\frac{3n+1}{2n+1}\left(\frac{1}{r_0^n \cos\alpha_1}\right)^{\frac{3n+1}{n(2n+1)}}\int_0^X (r_0^n \cos\alpha_1)^{\frac{1}{2n+1}}\,dX\right]^{\frac{(n+1)}{3n+1}}, \tag{37}$$

$$X = \int_0^{|x|} r_0(|x|)\,d|x|. \tag{38}$$

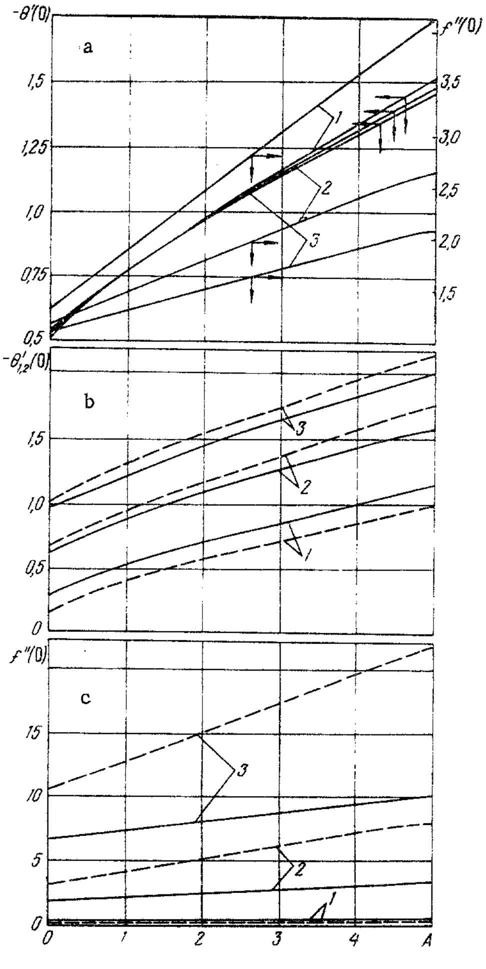

Fig. 1. Dependence of the heat exchange and friction characteristics on the curvature parameter. a: 1) n = 0.5; 2) 0.75; 3) 1.0; heat and mass exchange on the curvature parameter and the ratio Gr_2/Gr_1. b: 1) $Gr_2/Gr_1 = -0.9$; 2) 1; 3) 10 [solid curve) n = 1; dashes) n = 0.5] and friction on the curvature parameter and the ratio Gr_2/Gr_1. c: data the same as for b.

The heat- and mass-exchange coefficients are hence written, respectively, as

$$\mathrm{Nu}_i = -\Theta_i'(0)\left(\frac{2n+1}{3n+1}\right)^{\frac{n}{3n+1}} \mathrm{Gr}_1^{\frac{1}{2(n+1)}} \mathrm{Pr}_1^{\frac{n}{3n+1}} \frac{(r_0^n \cos\alpha_1)^{\frac{1}{2n+1}}}{\left[\int\limits_0^{|x|} r_0^{\frac{3n+1}{2n+1}} (\cos\alpha_1)^{\frac{1}{2n+1}} d|x|\right]^{\frac{n}{3n+1}}}, \quad i=1,2.$$

In particular cases:

1) sphere, $\cos\alpha_1 = \sin x$, $r_0 = \sin x$;

2) cone, $\cos\alpha_1 = \mathrm{const}$, $r_0 = x\sin\alpha_1$;

3) three-dimensional stagnation point, $\cos\alpha_1 = x$, $r_0 = x$;

4) vertical cylinder, $\cos\alpha_1 = 1$, $r_0 = 1$.

Plane problems of free convection of a binary mixture of a non-Newtonian fluid (particularly free convection around: 1) a vertical plate; 2) a horizontal cylinder; 3) a two-dimensional stagnation point; 4) a wedge) reduce to the system (30)-(32) (the case A = 0). In this case, the introduction of new variables proposed in [1] with the quantity x replaced by |x| therein is necessary.

NOTATION

x', y'	are the dimensional coordinates;
u', v'	are the dimensional velocities;
r	is the dimensional radial coordinate;
T	is the absolute temperature;
T_0	is the absolute temperature at the wall;
T_∞	is the absolute temperature as $y' \to \infty$;
C_0	is the concentration at the wall;
C_∞	is the concentration as $y' \to \infty$;
λ	is the heat conductivity;
β	is the coefficient of thermal volume expansion;
L	is the characteristic length;
g	is the free fall acceleration;
k	is the consistency coefficient;
n	is the non-Newtonian behavior parameter;
Ψ	is the modified stream function;
$\mathrm{Pr} = \dfrac{\rho c_p[Lg\beta(T_0 - T_\infty)]^{1/2}L}{\lambda\,\mathrm{Gr}^{\frac{1}{n+1}}}$	is the modified Prandtl number
$\mathrm{Gr}_1 = \left(\dfrac{\rho L^n}{k}\right)^2 [Lg\beta(T_0 - T_\infty)]^{1/2}$;	
$\mathrm{Gr}_2 = \left(\dfrac{\rho L^n}{k}\right)^2 [Lg\beta_1(C_0 - C_\infty)]^{1/2}$	is the modified Grasshopf number;
r_0	is the radius of the body of revolution;
η, $f(\eta)$, $\Theta_{1,2}(\eta)$	are the self-similar variables;
A	is the curvature parameter;
Nu	is the Nusselt number;
cf	is the friction coefficient.

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